Bifurcations In the Geometry of the Attractor

In Border Collision Normal Form

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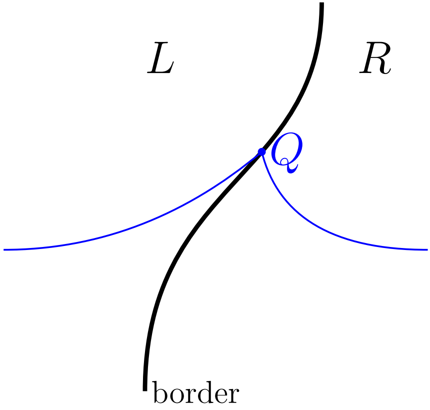
From Piecewise Smooth to Border Collison



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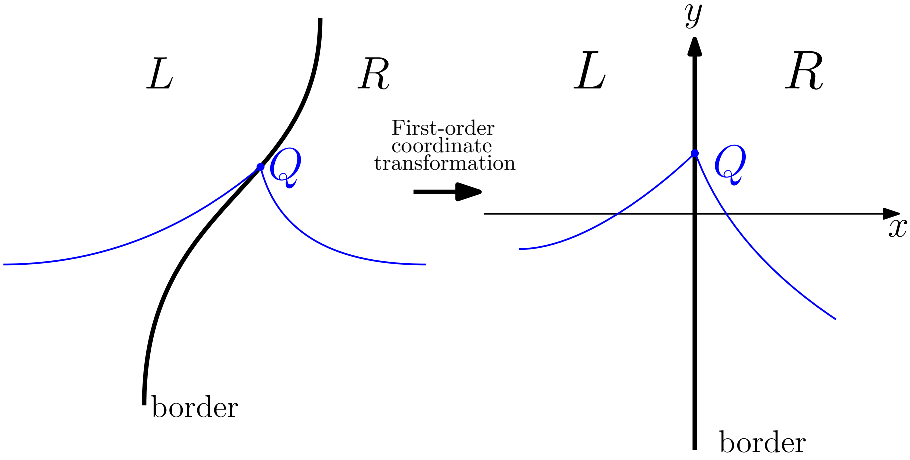

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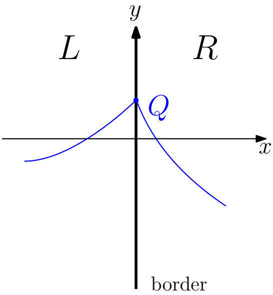

Border Collision Normal Form:

$$F : \mathbb{R}^2 \to \mathbb{R}^2$$
$$\mathbf{x_{n+1}} = A\mathbf{x_n} + \mathbf{m}$$

$$A = \begin{bmatrix} \tau_\alpha & 1 \\ -\delta_\alpha & 0 \end{bmatrix}, \quad \alpha = L, R$$

$$\mathbf{m} = \begin{bmatrix} \mu \\ 0 \end{bmatrix}, \quad \mathbf{x}_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

μ : bifurcation parameter,
WLOG, $\mu = 1$.

Some Definitions

Absorbing Region

A closed connected area $\mathcal{A}$ such that $T(\mathcal{A}) \subseteq \mathcal{A}$.
When $T(\mathcal{A}) = \mathcal{A}$, $\mathcal{A}$ is an invariant absorbing area.
[continuous map $T : \mathbb{R}^2 \to \mathbb{R}^2$]

Attractor

An invariant absorbing area $\mathcal{A}$ such that $\exists\, \mathcal{U} \supset \mathcal{A}$ open, we have $\lim_{n\to\infty} T^n(\mathcal{U}) = \mathcal{A}$ and the orbits of T are dense.

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attractor $\Rightarrow$ invariant absorbing area

Constructing Attractors

Non-attractor examples

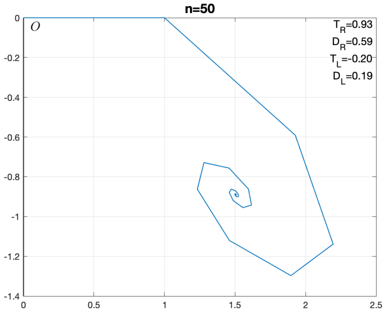


Figure: Fixed point

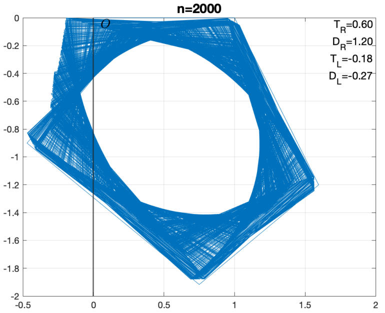


Figure: Periodic orbits

Constructing Attractors

Markov partition (simple, polygonal)

Conditions:

- $\mu > 0$,
- $F^n(O) = P_n = (0, y_0)$, where $y_0 < -\mu$,
- All points $P_1 P_2 ... P_{n-1}$ are on the right,
- Fixed point A is inside the triangle $P_1 P_{n+1} Q$,
- Q is in $P_1 P_2 ... P_n P_{n+1}$.

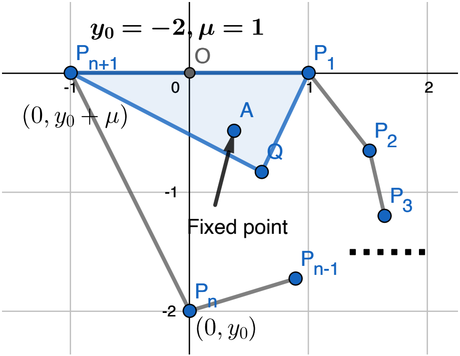
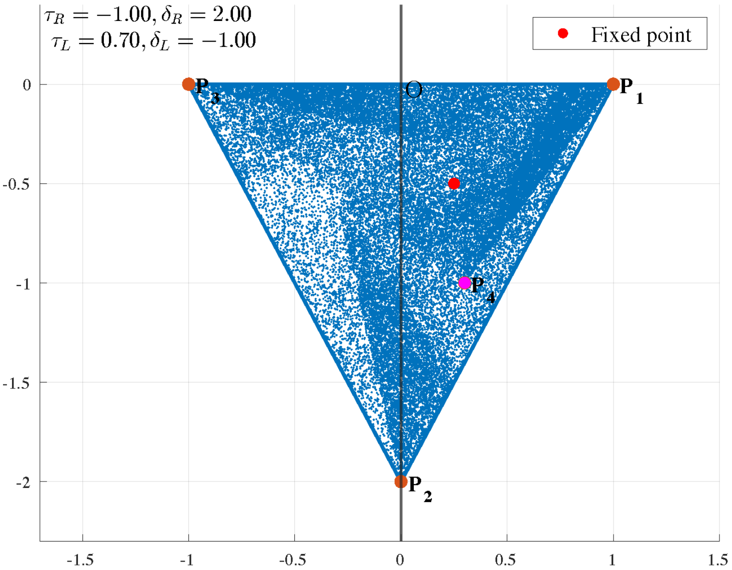


Figure: Markov partition

Constructing Attractors



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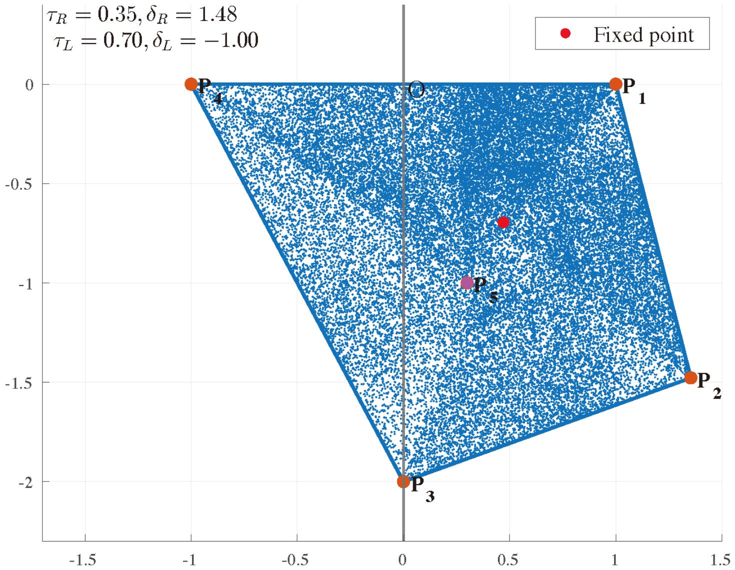

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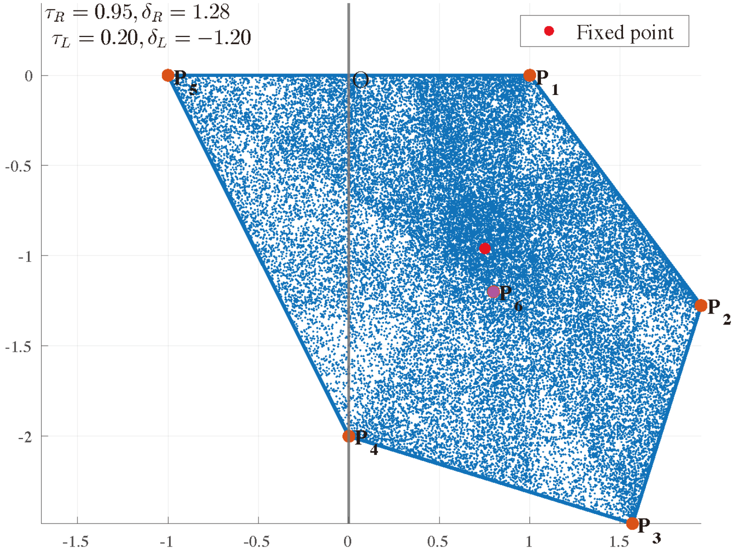

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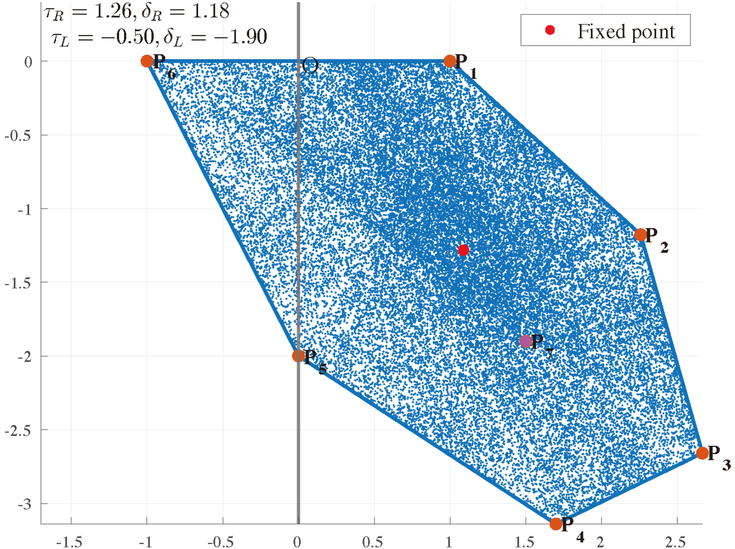

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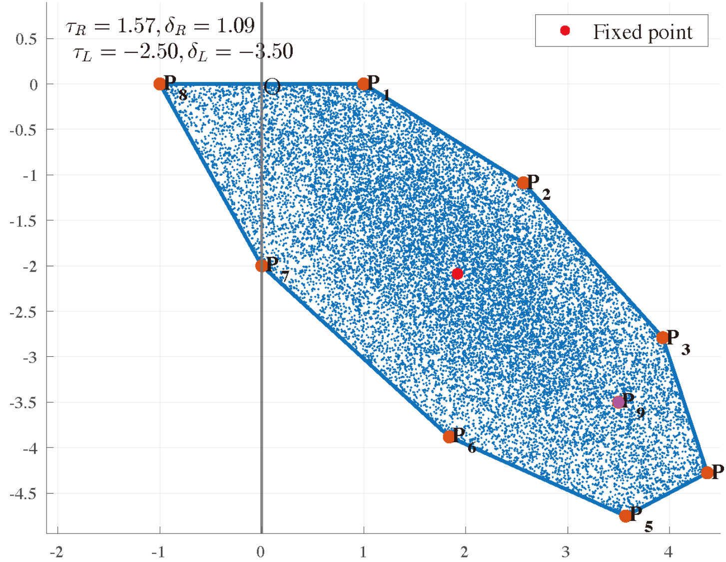

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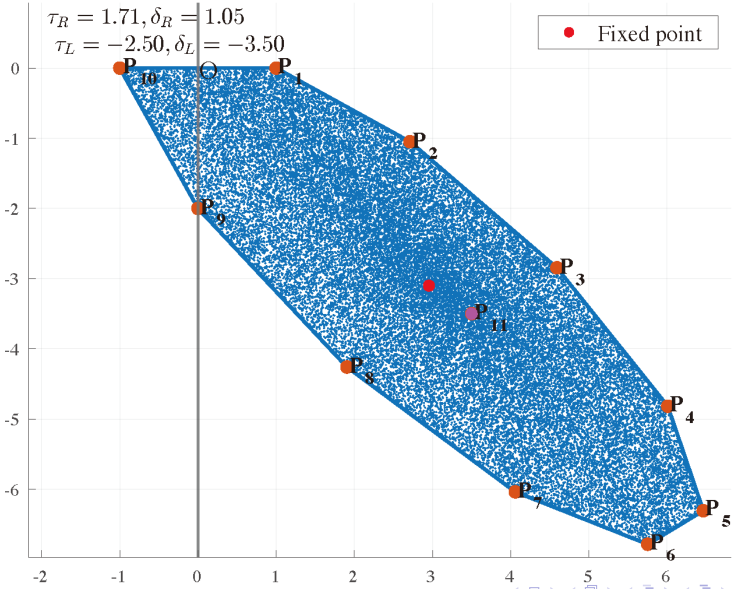
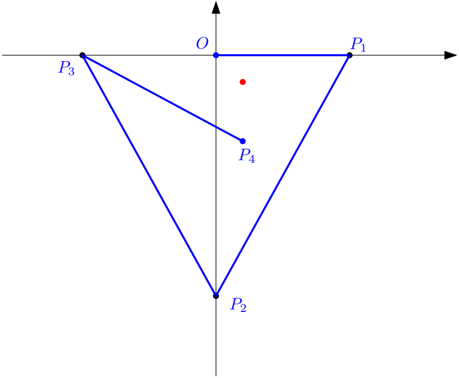
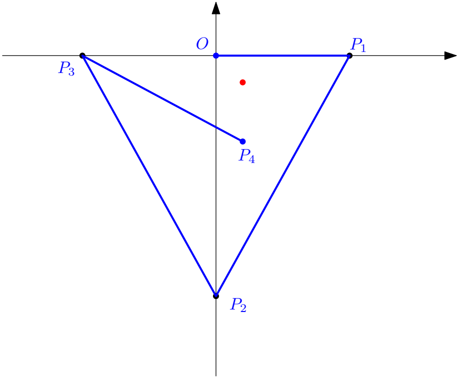




Two kinds of perturbations ⇒
Two kinds of bifurcations

- which side P_2 is on?
  (Determine τ_R, δ_R)
- Is P_4 in or out of the triangle?
  (Determine τ_L, δ_L)



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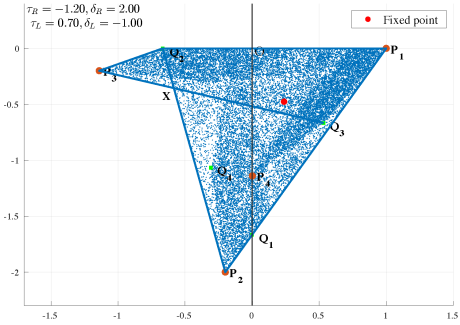
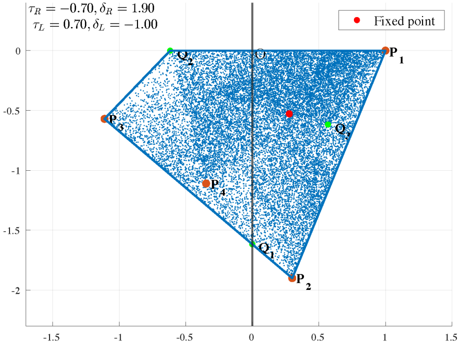
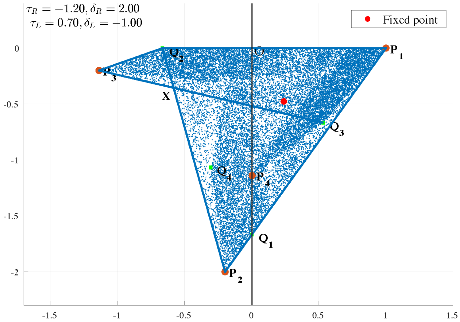
Continuity of the map ensures the existence of attractor

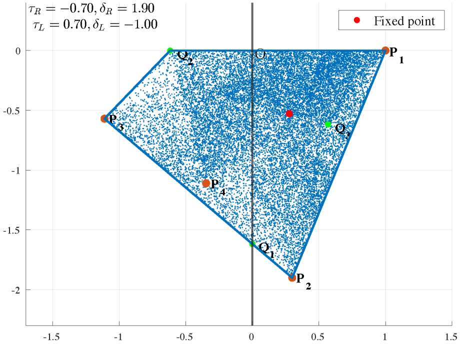
Bifurcations

Perturbation









what are the new boundaries?

Given a two-dimensional non-invertible differentiable map $T : \mathbb{R}^2 \to \mathbb{R}^2$, an invariant absorbing area A.

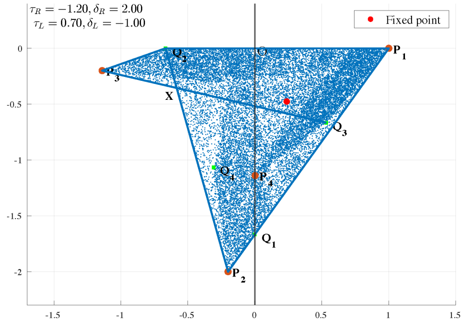
Critical Curve LC

Locus of points having two or more coincident preimages.

Define $\gamma = A \cap LC$, the boundary of the ∂A of the absorbing area A satisfies

$$\partial A \subset \bigcup_{k=1}^{m} T^k(\gamma)$$

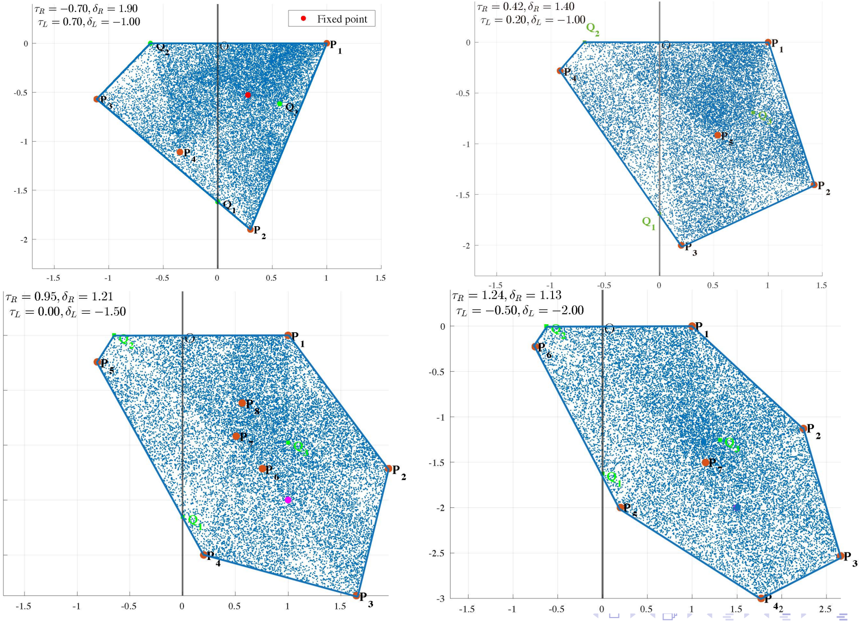
for some integer m.



Images of OQ_1:

$$OQ_1 \to P_1Q_2 - \left[\begin{array}{l} OP_1 \to P_1P_2 - \left[\begin{array}{l} P_1Q_1 \to P_2Q_2 \to P_3Q_3 \\ P_2Q_1 \to P_3Q_2 \to P_4Q_3 \end{array} \right. \\ OQ_2 \to P_1Q_3 \to P_2Q_4 \end{array} \right.$$

Bifurcations



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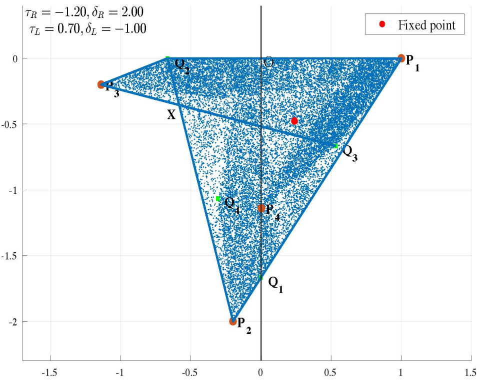

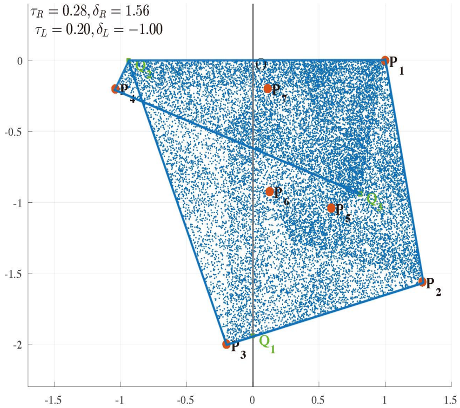

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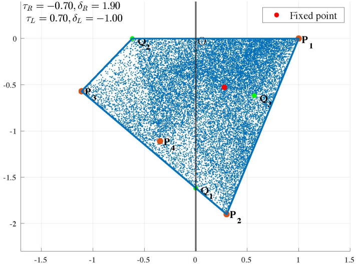

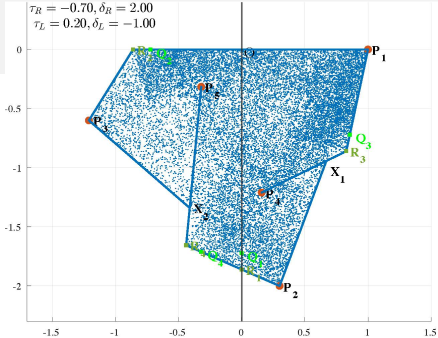

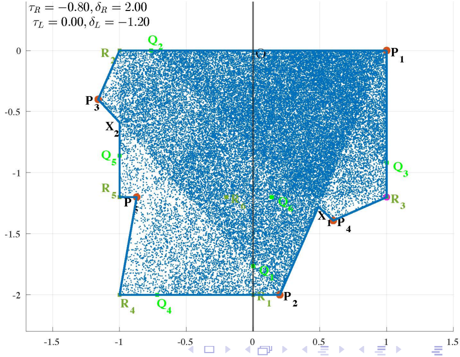

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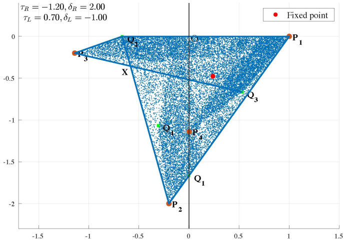

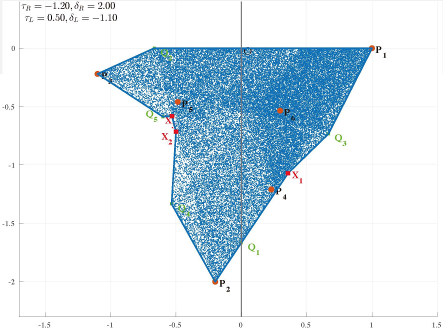

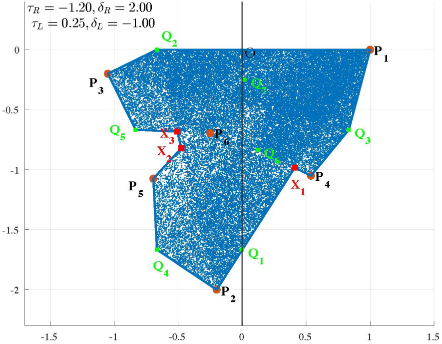

Conclusion

- small neighbourhood perturbation
- piecewise continuous map
- finite many sides

$\Rightarrow$ New attractors